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On a Uniform Method for Integration, Differentiation and Interpolation with Applications to the Numerical Solution of a Class of Partial Differential Equations

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Using a method analogous to that given in [1], we establish elementary analytical equalities giving rise to some useful applications; among them a method for the numerical integration of some partial differential equations.

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Let Ω be the rectangle $[a_1, a_2] \times [b_1, b_2]$, and

$$x_i = a_1 + ih, \qquad y_j = b_1 + jk,$$
$$h = (a_2 - a_1)/(S_1 + 1), \qquad k = (b_2 - b_1)/(S_2 + 1).$$

Each point $(x^*, y^*) \in \Omega$ divides Ω into four subdomains (rectangles):

$$\Omega_*{}^1 = \{(x, y) \in \Omega; x < x^*, y < y^*\}, \qquad \Omega_*{}^2 = \{(x, y) \in \Omega; x > x^*, y < y^*\},$$
$$\Omega_*{}^3 = \{(x, y) \in \Omega; x > x^*, y > y^*\}, \qquad \Omega_*{}^4 = \{(x, y) \in \Omega; x < x^*, y > y^*\}.$$

We shall denote also by Ω_{ij}^α ($\alpha = 1, 2, 3, 4$) the four rectangles analogous to the above ones, obtained by dividing Ω by means of the point (x_i, y_j).

Finally, let

$$A_1 = (a_1, b_1), \qquad A_2 = (a_2, b_1), \qquad A_3 = (a_2, b_2), \qquad A_4 = (a_1, b_2),$$
$$X_{1i} = X_{2i} = (x_i, b_1), \qquad X_{3i} = X_{4i} = (x_i, b_2),$$
$$Y_{1i} = Y_{4i} = (a_1, y_i), \qquad Y_{2j} = Y_{3j} = (a_2, y_j), \qquad X_{ij} = (x_i, y_j),$$
$$H_{ij}^1(x, y) = H(x_i - x)\, H(y_j - y), \qquad (x, y) \in \Omega_{ij}^1,$$
$$H_{ij}^2(x, y) = -H(x - x_i)\, H(y_j - y), \qquad (x, y) \in \Omega_{ij}^2,$$
$$H_{ij}^3(x, y) = H(x - x_i)(y - y_j), \qquad (x, y) \in \Omega_{ij}^3,$$
$$H_{ij}^4(x, y) = -H(x_i - x)\, H(y - y_j), \qquad (x, y) \in \Omega_{ij}^4,$$

H being the usual Heaviside function, and $\varphi \in \mathscr{C}^{r+s}(\Omega)$.

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Then, using integration by parts, we have:

(a) for $r > p$, $s > q$:

$$\iint_{\Omega_{ij}^{\alpha}} \frac{(x_i - x)^p}{p!} \frac{(y_j - y)^q}{q!} H_{ij}^{\alpha}(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r \partial y^s} \, dx \, dy$$

$$= \sum_{a=0}^{p} \sum_{b=0}^{q} \left[\frac{(x_i - x)^{p-a}}{(p-a)!} \frac{(y_j - y)^{q-b}}{(q-b)!} \frac{\partial^{r+s-a-b-2}\varphi}{\partial x^{r-a-1} \partial y^{s-b-1}} \right]_{A_\alpha}$$

$$- \sum_{a=0}^{p} \left[\frac{(x_i - x)^{p-a}}{(p-a)!} \frac{\partial^{r+s-q-a-2}\varphi}{\partial x^{r-a-1} \partial y^{s-q-1}} \right]_{Y_{\alpha j}}$$

$$- \sum_{b=0}^{q} \left[\frac{(y_j - y)^{q-b}}{(q-b)!} \frac{\partial^{r+s-p-b-2}\varphi}{\partial x^{r-p-1} \partial y^{s-b-1}} \right]_{X_{\alpha i}} + \left[\frac{\partial^{r+s-p-q-2}\varphi}{\partial x^{r-p-1} \partial y^{s-q-1}} \right]_{X_{ij}}; \tag{1}$$

(b) for $r = p$, $s = q$,

$$\iint_{\Omega_{ij}^{1}} \frac{(x_i - x)^p}{p!} \frac{(y_j - y)^q}{q!} H_{ij}^{1}(x, y) \frac{\partial^{p+q}\varphi}{\partial x^p \partial y^q} \, dx \, dy$$

$$= \sum_{a=0}^{p} \sum_{b=0}^{q} \frac{(x_i - a_1)^p}{p!} \frac{(y_j - b_1)^q}{q!} \frac{\partial^{p+q-a-b-2}\varphi}{\partial x^{p-a-1} \partial y^{q-b-1}} (a_1, b_1)$$

$$- \sum_{a=0}^{p-1} \int_{b_1}^{y_j} \frac{(x_i - a_1)^{p-a}}{(p-a)!} \frac{\partial^{p-a-1}\varphi}{\partial x^{p-a-1}} (a_1, y) \, dy$$

$$- \sum_{b=0}^{q-1} \int_{a_1}^{x_i} \frac{(y_j - b_1)^{q-b}}{(q-b)!} \frac{\partial^{q-b-1}\varphi}{\partial y^{q-b-1}} (x, b_1) \, dx + \iint_{\Omega_{ij}^{1}} \varphi(x, y) \, dx \, dy. \tag{2}$$

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Consider now the polynomials

$$P_{ij}(x_i - x, y_j - y) = P_{ij}(x, y) = \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq}^{ij} \frac{(x_i - x)^p}{p!} \frac{(y_j - y)^q}{q!},$$

and the functions

$$\Phi_{\lambda,\mu}^{\alpha}(x, y) = \sum_{(x_i, y_j) \in \Omega} P_{ij}(x, y) H_{ij}^{\alpha}(x, y) + \frac{(x^* - x)^\lambda}{\lambda!} \frac{(y^* - y)^\mu}{\mu!}. \tag{3}$$

Supposing $r > m$, $s > n$, we have:

$$\psi^{\alpha}(i, j; r, s; \varphi) = \iint_{\Omega} P_{ij}(x, y)\, H_{ij}^{\alpha}(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r\, \partial y^s}(x, y)\, dx\, dy$$

$$= \sum_{a=0}^{m} \sum_{b=0}^{n} \left[\frac{\partial^{a+b} P_{ij}(x, y)}{\partial x^a\, \partial y^b} \frac{\partial^{r+s-a-b-2}\varphi}{\partial x^{r-a-1}\, \partial y^{s-b-1}}(x, y)\right]_{A_\alpha}$$

$$- \sum_{a=0}^{m} \left[\frac{\partial^{a+n} P_{ij}(x, y)}{\partial x^a\, \partial y^n} \frac{\partial^{r+s-a-n-2}\varphi}{\partial x^{r-a-1}\, \partial y^{s-n-1}}(x, y)\right]_{Y_{\alpha i}}$$

$$- \sum_{b=0}^{n} \left[\frac{\partial^{m+b} P_{ij}(x, y)}{\partial x^m\, \partial y^b} \frac{\partial^{r+s-m-b-2}\varphi}{\partial x^{r-m-1}\, \partial y^{s-b-1}}(x, y)\right]_{X_{\alpha i}}$$

$$+ a_{pq}^{ij} \left[\frac{\partial^{r+s-p-q-2}\varphi(x, y)}{\partial x^{r-p-1}\, \partial y^{s-q-1}}\right]_{X_{ij}}, \tag{4}$$

and further

$$\iint_{\Omega} \Phi_{\lambda\mu}^{\alpha}(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r\, \partial y^s}(x, y)\, dx\, dy = \sum_{(x_i, y_j)\in\Omega} \psi^{\alpha}(i, j; r, s; \varphi)$$

$$+ \iint_{\Omega} \frac{(x^* - x)^{\lambda}}{\lambda!} \frac{(y^* - y)^{\mu}}{\mu!} H_x^{\alpha}(x, y) \frac{\partial^{r+s}\varphi(x, y)}{\partial x^r\, \partial y^s}\, dx\, dy. \tag{5}$$

3. Cubature Formula

Taking in (5)

$$x^* = a_2, \quad y^* = b_2, \quad \lambda = r, \quad \mu = s, \quad m < r, \quad n < s,$$

we obtain

$$\iint_{\Omega} \varphi(x, y)\, dx\, dy = \mathscr{F}(\varphi) - \sum_{(x_i, y_j)\in\Omega} \psi^{(1)}(i, j; r, s; \varphi)$$

$$+ \iint_{\Omega} \Phi_{\lambda\mu}^{1}(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r\, \partial y^s}(x, y)\, dx\, dy, \tag{6}$$

where

$$\mathscr{F}(\varphi) = \sum_{a=0}^{r-1} \sum_{b=0}^{s-1} \frac{(a_2 - a_1)^{r-a}}{(r-a)!} \frac{(b_2 - b_1)^{s-b}}{(s-b)!} \frac{\partial^{r+s-a-b-2}\varphi}{\partial x^{r-a-1}\, \partial y^{s-b-1}}(a_1, b_1)$$

$$- \sum_{a=0}^{r-1} \int_{b_1}^{b_2} \frac{(a_2 - a_1)^{r-a}}{(r-a)!} \frac{\partial^{r-a-1}\varphi}{\partial x^{r-a-1}}(a_1, y)\, dy$$

$$- \sum_{b=0}^{s-1} \int_{a_1}^{a_2} \frac{(b_2 - b_1)^{s-b}}{(s-b)!} \frac{\partial^{s-b-1}\varphi}{\partial y^{s-b-1}}(x, b_1)\, dx. \tag{7}$$

Denote now

$$\mathscr{R}(\varphi) = \iint_{\Omega} \Phi_{\lambda\mu}(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r \, \partial y^s} (x, y) \, dx \, dy. \tag{8}$$

We shall prove that we can choose the coefficients a_{pq}^{ij} so that $\mathscr{R}(\varphi)$ is, in absolute value, sufficiently small; then

$$\iint_{\Omega} \varphi(x, y) \, dx \, dy = \mathscr{F}(\varphi) - \sum_{(x_i, y_j) \in \Omega} \psi^1(i, j; r, s; \varphi) \tag{9}$$

can be considered as an approximate cubature formula for functions $\varphi(x, y)$ on Ω, with the remainder given by (8). The following theorem justifies it:

THEOREM. *If* $| \partial^{r+s}\varphi/\partial x^r \partial y^s | \leqslant M$ *in* Ω, *then we can choose* Φ^1, *i.e., the coefficients* a_{pq}^{ij}, *so that*

$$| \mathscr{R}(\varphi)| \leqslant A(h^2 + k^2)^{\frac{1}{2}t}, \qquad t = \min(m + 1, n + 1).$$

Proof. Denoting

$$I_{ij} = [a_1 + (i - 1)\, h, a_1 + ih] \times [b_1 + (j - 1)\, k, b_1 + jk],$$

we have

$$| \mathscr{R}(\varphi)| \leqslant M \sum_{(x_i, y_j) \in \Omega} \iint_{I_{ij}} | \Phi_{r,s}^1(x, y)| \, dx \, dy.$$

Notice that in I_{ij} we have

$$\Phi_{\lambda\mu}^1(x, y) = \sum_{u>i} \sum_{v>j} P_{uv}(x, y) \, H_{uv}^1(x, y) + \frac{(a_2 - x)^r}{r!} \frac{(b_2 - y)^s}{s!} H_*^1(x, y).$$

Put now

$$Q_{ij}(x, y) = \sum_{u>i} \sum_{v>j} P_{uv}(x, y) \, H_{uv}^1(x, y).$$

It is obvious that, in I_{ij}, Q_{ij} is a polynomial of degree m in x and n in y, and that, given the P_{ij}, the Q_{ij} can immediately be calculated. Conversely, if the Q_{ij} are given, one can easily calculate recursively the P_{ij}; namely we have:

$$P_{S_1 S_2} = Q_{S_1 S_2},$$

$$P_{S_1-i, S_2} = Q_{S_1-i, S_2} - \sum_{l=0}^{i-1} P_{S_1-l, S_2},$$

$$P_{S_1-i, S_2-j} = Q_{S_1-i, S_2-j} - \sum_{l=0}^{i-1} \sum_{t=0}^{j-1} P_{S_1-l, S_2-t}.$$

Choose now the Q_{ij} so that the function $\Phi_1^{rs}(x, y)$ and all its derivatives up to the mth in x and the nth in y are zero at a point $(\xi, \eta) \in I_{ij}$. It follows then that for $(x, y) \in I_{ij}$,

$$|\Phi_{rs}^1(x, y)| = O((h^2 + k^2)^{\frac{1}{2}a}), \qquad a = \min(m + 1, n + 1).$$

It follows that

$$\iint_{I_{ij}} |\Phi_{rs}^1(x, y)|\, dx\, dy = O((k^2 + k^2)^{\frac{1}{2}(a+2)}),$$

and

$$\mathscr{R}(\varphi) = O((h^2 + k^2)^{\frac{1}{2}a}) \tag{10}$$

is the global error for the formula (9). But if in the calculation of $\mathscr{F}(\varphi)$ there is an error of order $O((h^2 + k^2)^{\frac{1}{2}b})$, then obviously the global error will be of order $\min(a, b)$.

4. Interpolation and Numerical Differentiation Formulas

We return now to (5), and take $r > \lambda \geqslant m$, $s > \mu \geqslant n$. Take also instead of Ω, one of the domains $\Omega_*{}^\alpha$. We obtain:

$$\iint_{\Omega_*{}^\alpha} \Phi_{\lambda\mu}^\alpha(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r\, \partial y^s}\, dx\, dy$$

$$= \sum_{(x_i, y_j) \in \Omega_*{}^\alpha} \psi^\alpha(i, j; r, s; \varphi) + \mathscr{G}_{\alpha,\lambda,\mu}(\varphi) + \frac{\partial^{r+s-\lambda-\mu-2}\varphi}{\partial x^{r-\lambda-1}\, \partial y^{s-\mu-1}}(x^*, y^*),$$

where

$$\mathscr{G}_{\alpha,\lambda,\mu}(\varphi) = \sum_{a=0}^{\lambda} \sum_{b=0}^{\mu} \left[\frac{(x^* - x)^{\lambda-a}}{(\lambda - a)!} \frac{(y^* - y)^{\mu-b}}{(\mu - b)!} \frac{\partial^{r+s-a-b-2}\varphi}{\partial x^{r-a-1}\, \partial y^{s-b-1}}\right]_{A_\alpha{}^*}$$

$$- \sum_{a=0}^{\lambda} \left[\frac{(x^* - x)^{\lambda-a}}{(\lambda - a)!} \frac{\partial^{r+s-a-\mu-2}\varphi}{\partial x^{r-a-1}\, \partial y^{s-\mu-1}}\right]_{Y_{\alpha,j}}$$

$$- \sum_{b=0}^{\mu} \left[\frac{(y^* - y)^{\mu-b}}{(\mu - b)!} \frac{\partial^{r+s-\lambda-b-2}\varphi}{\partial x^{r-\lambda-1}\, \partial y^{s-b-1}}\right]_{X_{\alpha i}}. \tag{11}$$

As before, we can choose the polynomial P_{ij}, i.e., the coefficients a_{pq}^{ij}, so that

$$\left|\iint_{\Omega_*{}^\alpha} \phi_{\lambda\mu}^1(x, y) \frac{\partial^{r+s}\varphi}{\partial x^r\, \partial y^s}(x, y)\right| = O((h^2 + k^2)^{\frac{1}{2}a}), \quad a = \min(m + 1, n + 1)$$

and obtain the approximate equality

$$\frac{\partial^{r+s-\lambda-\mu-2}\varphi}{\partial x^{r+\lambda-1}\,\partial y^{s-\mu-1}}(x^*, y^*) \simeq - \sum_{(x_i, y_j)\in\Omega_*^\alpha} \psi^\alpha(i, j; r, s; \varphi) + \mathscr{G}_{\alpha,\lambda\mu}(\varphi). \quad (12)$$

If $r = \lambda + 1$, $s = \mu + 1$, we obviously obtain an interpolation formula, while for $r > \lambda + 1$, $s > \mu + 1$, this is a numerical differentiation formula. This formula can be written in another form:

$$\frac{\partial^{u+v}\varphi}{\partial x^u\,\partial y^v}(x^*, y^*) \simeq - \sum_{(x_i, y_j)\in\Omega_*^\alpha} \psi^\alpha(i, j; m + 1, n + 1; \varphi) + \mathscr{G}_{\alpha,m-u,n-v}(\varphi),$$

from which we obtain

$$\frac{\partial^{u+v}\varphi}{\partial x^u\,\partial y^v}(x^*, y^*)$$
$$= -\frac{1}{4}\sum_{\alpha=1}^{4}\Bigg[\sum_{(x_i, y_j)\in\Omega_*^\alpha} \psi^\alpha(i, j; m + 1, n + 1; \varphi) + \mathscr{G}_{\alpha,m-\mu,n-v}(\varphi)\Bigg]. \quad (13)$$

5. Application to the Numerical Integration of a Class of Partial Differential Equations

Consider the equation

$$\mathscr{L}\varphi \equiv \sum_{p\leqslant m, q\leqslant n} a_{pq}(x, y)\frac{\partial^{m+n-p-q}\varphi}{\partial x^{m-p}\,\partial y^{n-q}} = f(x, y), \quad (14)$$

where a_{pq} are functions in $\mathscr{C}(\Omega)$, such that

$$a_{mn} = 1, \quad a_{m,i} = a_{jn} = 0, \qquad i < m, \quad j < n,$$

and the associated polynomials

$$P_{ij}(x, y) = \sum_{p=0}^{m}\sum_{q=0}^{n} a_{pq}(x_i, y_j)\frac{(x_i - x)^p}{p!}\frac{(y_j - y)^q}{q!}$$
$$= \sum_{p=0}^{m}\sum_{q=0}^{n} a_{pq}^{ij}\frac{(x_i - x)^p}{p!}\frac{(y_j - y)^q}{q!}.$$

With this notation, we obtain from (1), where we take $r = m + 1$, $s = n + 1$:

$$\iint_{\Omega^1_{ij}} P_{ij}(x, y)\, H^1_{ij}(x, y) \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy$$

$$= \sum_{a=0}^{m} \sum_{b=0}^{n} \Big\{ \Big[\frac{\partial^{a+b} P_{ij}}{\partial x^a\, \partial y^b} \frac{\partial^{m+n-a-b}\varphi}{\partial x^{m-a}\, \partial y^{n-b}} \Big]_{A_1} - \Big[\frac{\partial^{a+b} P_{ij}}{\partial x^a\, \partial y^b} \frac{\partial^{m+n-a-b}\varphi}{\partial x^{m-a}\, \partial y^{n-b}} \Big]_{Y_{1j}}$$

$$- \Big[\frac{\partial^{a+b} P_{ij}}{\partial x^a\, \partial y^b} \frac{\partial^{m+n-a-b}\varphi}{\partial x^{m-a}\, \partial y^{n-b}} \Big]_{X_{1i}} \Big\} + (\mathscr{L}\varphi)(x_i , y_j)$$

$$= \mathscr{T}_{ij}(\varphi) + (\mathscr{L}\varphi)(x_i , y_j), \tag{15}$$

the functional $\mathscr{T}_{ij}$ being defined by (15).

On the other hand we have, for $x^* = x_{N_1+1}$, $y^* = y_{N_2+1}$:

$$\iint_{\Omega^1_{N_1+1,N_2+1}} \frac{(x_{N_1+1} - x)^{m-\lambda}}{(m - \lambda)!} \frac{(y_{N_2+1} - y)^{n-\mu}}{(n - \mu)!} \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy$$

$$= \frac{\partial^{\lambda+\mu}\varphi}{\partial x^\lambda\, \partial y^\mu} (x_{N_1+1} , y_{N_2+1}) + \mathscr{H}_{\lambda\mu}(\varphi),$$

where

$$\mathscr{H}_{\lambda\mu}(\varphi) = \sum_{a=0}^{\lambda} \sum_{b=0}^{\mu} \Big[\frac{(x_{N_1+1} - x)^{m-\lambda-a}}{(m - \lambda - a)!} \frac{(y_{N_1+1} - y)^{n-\mu-b}}{(n - \mu - b)!} \frac{\partial^{m+n-a-b}\varphi}{\partial x^{m-a}\, \partial y^{n-b}} \Big]_{A_1}$$

$$- \sum_{a=0}^{\lambda} \Big[\frac{(x_{N_1+1} - x)^{m-\lambda-a}}{(m - \lambda - a)!} \frac{\partial^{m+\mu-a}\varphi}{\partial x^{m-a}\, \partial y^{\mu}} \Big]_{Y_{1j}}$$

$$- \sum_{b=0}^{\mu} \Big[\frac{(y_{N_2+1} - y)^{n-\mu-b}}{(n - \mu - b)!} \frac{\partial^{n+\lambda-b}\varphi}{\partial x^{\lambda}\, \partial y^{m-b}} \Big]_{X_{1i}} .$$

If we consider the function:

$$\Psi_{\lambda\mu}(x, y) = \sum_{i \leqslant N_1, j \leqslant N_2} \sigma^{ij}_{\lambda\mu} P_{ij}(x, y)\, H^1_{ij}(x, y)$$

$$+ \frac{(x_{N_1+1} - x)^\lambda}{\lambda!} \frac{(y_{N_2+1} - y)^\mu}{\mu!} H_*(x, y), \tag{16}$$

the σ_{pq}^{ij} being constants to be determined, we have

$$\iint_{\Omega^1_{N_1+1,N_2+1}} \Psi_{\lambda\mu}(x, y) \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy$$

$$= \sum_{\substack{i\leqslant N_1\\ j\leqslant N_2}} \sigma_{\lambda\mu}^{ij}(\mathscr{L}u)(x_i, y_j) + \frac{\partial^{\lambda+\mu}\varphi}{\partial x^\lambda\, \partial y^\mu}(x_{N_1+1}, y_{N_2+1}) + \mathscr{J}_{\lambda\mu}(\varphi), \tag{17}$$

where:

$$\mathscr{J}_{\lambda\mu}(\varphi) = \mathscr{H}_{\lambda\mu}(\varphi) + \sigma_{\lambda\mu}^{ij}\mathscr{T}_{ij}(\varphi). \tag{18}$$

The functionals $\mathscr{H}_{\lambda\mu}$ and $\mathscr{T}_{ij}$ involve only values of φ and its derivatives up to the $(m+n)$th order, taken only on the sides $x = a_1$ and $y = b_1$ of Ω, and at points $(x_i, y_j) \in \Omega$, with $i \leqslant N_1, j \leqslant N_2$.

Taking into account (14), we obtain from (17):

$$\frac{\partial^{\lambda+\mu}\varphi}{\partial x^\lambda\, \partial y^\mu}(x_{N_1+1}, y_{N_2+1}) = \iint_{\Omega^1_{N_1+1,N_2+1}} \Psi_{\lambda\mu}(x, y) \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy$$

$$- \sum_{\substack{i\leqslant N_1\\ j\leqslant N_2}} \sigma_{\lambda\mu}^{ij} f(x_i, y_j) - \mathscr{J}_{\lambda\mu}(\varphi).$$

We will prove now that we can choose the coefficients $\sigma_{\lambda\mu}^{ij}$ so that

$$\iint_{\Omega_{N_1+1,N_2+1}} \Psi_{\lambda\mu}(x, y) \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy = O((h^2 + k^2)^{1/2}). \tag{19}$$

With that choice, the equalities $(\lambda = 0, 1,..., m-1, \mu = 0, 1,..., n-1)$

$$\frac{\partial^{\lambda+\mu}\varphi}{\partial x^\lambda\, \partial y^\mu}(x_{N_1+1}, y_{N_2+1}) \simeq - \sum_{\substack{i\leqslant N_1\\ j\leqslant N_2}} \sigma_{\lambda\mu}^{ij} f(x_i, y_j) - \mathscr{J}_{\lambda\mu}(\varphi) \tag{20}$$

will give approximate numerical values of the derivatives of φ in (x_{N_1+1}, y_{N_2+1}). The following theorem states the situation more precisely; for the sake of simplicity, we take $h = k$.

THEOREM. *If*

(a) $\varphi \in \mathscr{C}^{m+n+2}(\Omega)$,

(b) *the coefficients* $a_{pq}(\xi, \eta) \in \mathscr{C}(\Omega)$,

(c) *the constants* $\sigma_{ij}^{\lambda\mu}$ *are chosen so that*

$$\Psi_{\lambda\mu}(x_i, y_j) = 0, \tag{21}$$

(d) $MW < 1$,

where M is the least upper bound of all the derivatives of P_{ij} in Ω, and

$$W = V_1 + V_2 , \qquad V_1 = \frac{N_1 N_2 h^2}{\lambda\mu}, \qquad V_2 = 2 \max\left(\frac{N_1 h}{\lambda}, \frac{N_2 k}{\mu}\right), \tag{22}$$

then

$$| \sigma_{\lambda\mu}^{N_1-i, N_2-j} | \leqslant K \frac{[(i+1)^\lambda - i^\lambda]}{\lambda!} \frac{[(j+1)^\mu - j^\mu]}{\mu!} h^{\lambda+\mu}, \tag{23}$$

K being a constant satisfying

$$K > 1/(1 - MW). \tag{23'}$$

Proof. Taking in (21) $i = N_1 + 1$, or $j = N_2 + 1$, we obtain:

$$\sigma_{\lambda\mu}^{N_1+i,j} = \sigma_{\lambda\mu}^{i,N_2+1} = 0,$$

and, for $i = N_1$, $j = N_2$,

$$\sigma_{\lambda\mu}^{N_1 N_2} = -h^{\lambda+\mu}/\lambda!\,\mu!\,,$$

which is in accordance with (23). Using induction suppose now that (23) is satisfied for $i \leqslant s$, $j < t$; we will prove that (23) is true also for $i = s$, $j = t$. To this end, consider:

$$\Psi_{\lambda\mu}(x_{N_1-s}, y_{N_2-t}) = \sigma_{\lambda\mu}^{N_1-s, N_2-t} + \sum_{\substack{i\leqslant s\\ j<t}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} P_{N_1-i, N_2-j}(x_{N_1-s}, y_{N_2-t})$$
$$+ \frac{(x_{N_1+1} - x_{N_1-s})^\lambda}{\lambda!} \frac{(y_{N_2+1} - y_{N_2-t})^\mu}{\mu!} = 0,$$

which can also be written as

$$-\sum_{\substack{i\leqslant s\\ j\leqslant t}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} = \sum_{\substack{i\leqslant s\\ j\leqslant t}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} [P_{N_1-i, N_2-j}(x_{N_1-s}, y_{N_2-t}) - 1]$$
$$+ \frac{(s+1)^\lambda (t+1)^\mu}{\lambda!\,\mu!} h^{\lambda+\mu}. \tag{24}$$

It is easy to see that

$$\sum_{\substack{i\leqslant s\\ j\leqslant t}} = \sum_{\substack{i=s\\ j=t}} + \sum_{\substack{i\leqslant s-1\\ j\leqslant t}} + \sum_{\substack{i\leqslant s\\ j\leqslant t-1}} - \sum_{\substack{i\leqslant s-1\\ j\leqslant t-1}},$$

and, by means of analogs of (24), one gets:

$$\sum_{\substack{i \leqslant s-1 \\ j \leqslant t}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} = - \sum_{\substack{i \leqslant s-1 \\ j \leqslant t}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} [P_{N_1-i, N_2-j}(x_{N_1-s-1}, y_{N_2-t}) - 1] - \frac{s^\lambda (t+1)^\mu}{\lambda!\, \mu!} h^{\lambda+\mu},$$

$$\sum_{\substack{i \leqslant s \\ j < t-1}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} = - \sum_{\substack{i \leqslant s \\ j \leqslant t-1}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} [P_{N_1-i, N_2-j}(x_{N_1-s}, y_{N_2-t-1}) - 1] - \frac{(s+1)^\lambda t^\mu}{\lambda!\, \mu!} h^{\lambda+\mu},$$

$$\sum_{\substack{i \leqslant s-1 \\ j \leqslant t-1}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} = - \sum_{\substack{i \leqslant s-1 \\ j \leqslant t-1}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} [P_{N_1-i, N_2-j}(x_{N_1-s-1}, y_{N_2-t-1}) - 1] - \frac{s^\lambda t^\mu}{\lambda!\, \mu!} h^{\lambda+\mu}.$$

It follows that

$$\sigma_{\lambda\mu}^{N_1-s, N_2-t} = \sum_{\substack{i \leqslant s-1 \\ j \leqslant t-1}} \sigma_{\lambda\mu}^{N_1-i, N_2-j} \left(\frac{\partial^2 P_{N_1-i, N_2-j}}{\partial x\, \partial y} \right)^* h^2 + \sum_{j \leqslant t-1} \sigma_{\lambda\mu}^{N_1-s, N_2-j} \left(\frac{\partial P_{N_1-s, N_2-j}}{\partial x} \right)^* h + \sum_{i \leqslant s-1} \sigma_{\lambda\mu}^{N_1-i, N_2-t} \left(\frac{\partial P_{N_1-i, N_2-t}}{\partial y} \right)^* h + \frac{[(s+1)^\lambda - s^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\, \mu!} h^{\lambda+\mu}$$

(the asterisk means that the partial derivative is taken at a suitable point of Ω). Majorizing the σ's on the right-hand side of the above equality by means of (23), we obtain:

$$| \sigma_{\lambda\mu}^{N_1-s, N_2-t} | \leqslant \frac{[(s+1)^\lambda - s^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\, \mu!} h^{\lambda+\mu} + MKh^{\lambda+\mu} \left[\sum_{\substack{i \leqslant s-1 \\ j \leqslant t-1}} \frac{[(i+1)^\lambda - i^\lambda][(j+1)^\mu - j^\mu]}{\lambda!\, \mu!} h^2 + \sum_{j \leqslant t-1} \frac{[(s+1)^\lambda - s^\lambda][(j+1)^\mu - j^\mu]}{\lambda!\, \mu!} h + \sum_{i \leqslant s-1} \frac{[(i+1)^\lambda - i^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\, \mu!} h \right]$$

$$\leqslant \frac{[(s+1)^\lambda - s^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\,\mu!} h^{\lambda+\mu}$$
$$+ MKh^{\lambda+\mu}\Big[\frac{s^\lambda t^\mu}{\lambda!\,\mu!} h^2$$
$$+ \frac{[(s+1)^\lambda - s^\lambda]\, t^\mu}{\lambda!\,\mu!} h + \frac{s^\lambda[(t+1)^\mu - t^\mu]}{\lambda!\,\mu!} h\Big].$$

Taking into account (22), we get:

$$\frac{s^\lambda t^\mu}{\lambda!\,\mu!} h^{\lambda+\mu+2} \leqslant V_2 \frac{s^{\lambda-1} t^{\mu-1}}{(\lambda-1)!\,(\mu-1)!} h^{\lambda+\mu}$$
$$\leqslant V_2 \frac{[(s+1)^\lambda - s^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\,\mu!} h^{\lambda+\mu},$$
$$\frac{s^\lambda[(t+1)^\mu - t^\mu]}{\lambda!\,\mu!} h^{\lambda+\mu+1} \leqslant V_1 \frac{[(s+1)^\lambda - s^\lambda][(t+1)^\mu - t^\mu]}{\lambda!\,\mu!} h^{\lambda+\mu},$$

and analogous inequalities. With this, the conclusion (23) follows.

6

We can finally prove the following

THEOREM. *If*

(a) *the coefficients $a_{pq}(x, y)$ are continuous in the domain Ω;*

(b) *for all $u \leqslant m + 1, v \leqslant n + 1$:*

$$\left| \frac{\partial^{u+v} P_{ij}(x, y)}{\partial x^u\, \partial y^v} \right| \leqslant M;$$

and

(c) *φ is a solution of* (14), *having all its partial derivatives up to the mth in x and the nth in y bounded, then the error* (19) *in the computation of u and its partial derivatives up to the* $(m-1)$*th in x and the* $(n-1)$*th in y is* $O(h)$.

Proof. Obviously, since

$$\left| \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}} \right| \leqslant \mathscr{R},$$

$$\left| \iint_{I_{ij}} \Psi_{\lambda\mu}(x, y) \frac{\partial^{m+n+2}\varphi}{\partial x^{m+1}\, \partial y^{n+1}}\, dx\, dy \right| \leqslant \mathscr{R} \iint_{I_{ij}} |\Psi_{\lambda\mu}(x, y)|\, dx\, dy;$$

also, since $\Psi_{\lambda\mu}(x_{N_1-i}, y_{N_2-j}) = 0$, taking into account hypothesis (b) and the estimate (23), we have

$$| \Psi_{\lambda\mu}(x, y)| \leqslant \left(\frac{s^{\lambda-1}t^{\mu}}{(\lambda - 1)!\, \mu!} + \frac{s^{\lambda}(t-1)^{\mu}}{\lambda!\, (\mu - 1)!}\right) h^{\lambda+\mu}$$

$$+ \frac{MK}{\lambda!\, \mu!} \sum_{\substack{i \leqslant s \\ j \leqslant t}} [[(i + 1)^{\lambda} - i^{\lambda}][(j + 1)^{\mu} - j^{\mu}]\, h^{\lambda+\mu+1}.$$

But

$$\left(\frac{s^{\lambda-1}t^{\mu}}{(\lambda - 1)!\, \mu!} + \frac{s^{\lambda}t^{\mu-1}}{\lambda!\, (\mu - 1)!}\right) h^{\lambda+\mu} = O(h)$$

and

$$\frac{1}{\lambda!\, \mu!} \sum_{\substack{i \leqslant s \\ j \leqslant t}} [(i + 1)^{\lambda} - i^{\lambda}][(j + 1)^{\mu} - j^{\mu}]\, h^{\lambda+\mu+1}$$

$$\leqslant \sum_{\substack{i \leqslant s \\ j \leqslant t}} \frac{(s + 1)^{\lambda}\, (t + 1)^{\mu}}{\lambda!\, \mu!}\, h^{\lambda+\mu+1} = O(h),$$

so that the conclusion of the theorem follows immediately.

Remarks. (1) It is obvious that by this method the values of φ can be calculated at all points of Ω, provided that φ and its partial derivatives are known on the sides $x = a_1$ and $y = b_1$.

(2) The method of numerical integration given in the last sections can be useful when one has to integrate a large number of equations (14) with the same left-hand side, because the coefficients $\sigma_{\lambda\mu}$ do not depend on the right-hand side of (14). A computer will have to compute these coefficients just once for all of the equations.

Reference

1. A. Haimovici, Sur une certaine approximation des distributions généralisant des formules d'intégration, de dérivation numérique et d'interpolation, *Rev. Roumaine Math. Pures Appl.*, **15** (1970), 1415–1420.